MTH 303
Real analysis **Homework 7**

1. Let $f : [a, b] \to \mathbb{R}$ be such that $m \leq f(x) \leq M$ for some $m, M \in \mathbb{R}$ and for all $x \in [a, b]$. Show that
$$m(b - a) \leq L(P, f) \leq U(P, f) \leq M(b - a)$$
for every partition P of $[a, b]$.

2. Let $f : [a, b] \to \mathbb{R}$ be a bounded function. If
$$U(P, f) - L(P, f) < \epsilon \tag{1}$$
for some $\epsilon > 0$ and some partition P of $[a, b]$, then show that the inequality (1) holds for every refinement of P.

3. (a) If $f, g \in \mathcal{R}([a, b])$ and $\alpha \in \mathbb{R}$, then $f + \alpha g \in \mathcal{R}([a, b])$ and
$$\int_a^b (f + \alpha g) = \int_a^b f + \alpha \int_a^b g.$$
(b) If $f \in \mathcal{R}([a, b])$ and $c \in (a, b)$, then $f \in \mathcal{R}([a, c])$ and $f \in \mathcal{R}([c, b])$. Moreover,
$$\int_a^b f = \int_a^c f + \int_c^b f.$$
(c) If $f, g \in \mathcal{R}([a, b])$ and if $f(x) \leq g(x)$ for all $x \in [a, b]$, then
$$\int_a^b f = \int_a^b g.$$
(d) If $f \in \mathcal{R}([a, b])$ and if $|f(x)| \leq M$ for some $M > 0$ and for all $x \in [a, b]$, then
$$\left| \int_a^b f \right| \leq M(b - a).$$
(e) If $f \in \mathcal{R}([a, b])$, then show that $f^+ = \max\{f, 0\}$ and $f^- = \min\{f, 0\}$ are Riemann integrable on $[a, b]$.
(f) If $f \in \mathcal{R}([a, b])$, then show that $f^2 \in \mathcal{R}([a, b])$.
(g) If $f, g \in \mathcal{R}([a, b])$, then show that $fg \in \mathcal{R}([a, b])$. Also, show that
$$\left| \int_a^b fg \right| \leq \left(\int_a^b f^2 \right)^{\frac{1}{2}} \left(\int_a^b g^2 \right)^{\frac{1}{2}}.$$
The above inequality is known as the Cauchy-Schwarz inequality for Riemann integrable functions.

4. If $f \in \mathcal{R}([a,b])$, then show that $|f| \in \mathcal{R}([a,b])$ and
$$\left|\int_a^b f\right| \leq \int_a^b |f|.$$
Also, show by an example that the converse of the above may fail.

5. Let $x_0 \in [a,b]$ and $f : [a,b] \to \mathbb{R}$ be defined by $f(x_0) = 1$ and $f(x) = 0$ for all $x \neq x_0$. Show that $f \in \mathcal{R}([a,b])$ and $\int_a^b f(x)dx = 0$.

6. Let $f : [0,1] \to \mathbb{R}$ be defined by $f(x) = 1$ for all $x \in [0,1)$ and $f(1) = 0$. Show that $f \in \mathcal{R}([0,1])$ and $\int_a^b f(x)dx = 1$.

7. Suppose $f \geq 0, f$ is continuous on $[a,b]$, and $\int_a^b f(x)dx = 0$. Prove that $f(x) = 0$ for all $x \in [a,b]$. Also, compare this with previous question.

8. Let f be continuous on $[a,b]$, and $\int_a^b f(x)dx = 0$, then prove that $f(x) = 0$ for at least one $x \in [a,b]$.

9. Let f be continuous on $[a,b]$, and $\int_a^b f(x)g(x)dx = 0$ for every continuous function g on $[a,b]$, then prove that $f(x) = 0$ for all $x \in [a,b]$.

10. Suppose f is a bounded real function on $[a,b]$, and $f^2 \in \mathcal{R}([a,b])$. Does it follow that $f \in \mathcal{R}([a,b])$? Does the answer change if we assume that $f^3 \in \mathcal{R}([a,b])$?

11. Suppose f is a bounded real function on $[a,b]$ and f has only finitely many points of discontinuity on $[a,b]$. Show that $f \in \mathcal{R}([a,b])$.
(We proved it in class for functions having discontinuity at a single point.)

12. Let C denote the Cantor set in $[0,1]$. Let f be a bounded real function on $[0,1]$ which is continuous at every point outside C. Show that $f \in \mathcal{R}([0,1])$.
(Hint: Recall that the Cantor set C has length zero and can be covered by finitely intervals whose total length can be made as small as needed. Use this and proceed as in the previous question.)

13. Consider the function f defined on $\mathbb{R}$ by $f(x) = 0$ if x is irrational and $f(x) = \frac{1}{n}$ if $x = \frac{m}{n}$, where m and n are integers without any common divisors. Is f Riemann integrable on $[0,1]$?

14. Let f and g be bounded functions on $[a,b]$ such that the composite $f \circ g$ is Riemann integrable on $[a,b]$. Does it imply that both f and g are Riemann integrable on $[a,b]$?

15. Show by an example that composition of two Riemann integrable functions may not be Riemann integrable.

16. Suppose f is a real function on $(0,1]$ and $f \in \mathcal{R}([t,1])$ for every $t > 0$. Define

$$\int_a^b f(x)dx = \lim_{t \to 0} \int_t^1 f(x)dx$$

if this limit exists and is finite.
(Note: The notation $\int_a^b f(x)dx$ on LHS should not be taken as integral of f in the sense of Riemann integral)

(a) If $f \in \mathcal{R}([0,1])$, then show that the above definition of integral agrees with the Riemann integral.

(b) Construct a function f such that the above limit exists, although it fails to exist with $|f|$ in place of f.

17. Suppose $f \in \mathcal{R}([a,b])$ for every $b > a$, where a is fixed. Define

$$\int_a^\infty f(x)dx = \lim_{b \to \infty} \int_a^b f(x)dx$$

if the limit exists and is finite. In that case, we say that the integral on the LHS converges. If it also converges after f has been replaced by $|f$, it is said to converge absolutely.

Assume that $f(x) \geq 0$ and that f decreases monotonically on $[1, \infty)$. Prove that the integral $\int_1^\infty f(x)dx$ converges if and only if the series $\sum_{n=1}^{\infty} f(n)$ converges.

18. Let $g : \mathbb{R} \to \mathbb{R}$ be differentiable. Let $F(x) = \int_0^{g(x)} t^2 dt$. Show that $F'(x) = g^2(x)g'(x)$ for all $x \in \mathbb{R}$.

19. Let $f \in \mathcal{C}^2([a,b])$. Show that

$$\int_a^b f''(x)dx = (bf'(b) - f(b)) - (af'(a) - f(a)).$$

(Hint: Integration by parts).

20. (Mean value property for integrals): Let f be continuous real function on $[a,b]$. Show that there exists $c \in [a,b]$ such that

$$\int_a^b f(x)dx = f(c)(b-a).$$

21. Let f be monotone function on $[a,b]$. Show that there exists $c \in [a,b]$ such that

$$\int_a^b f(x)dx = f(a)(c-a) + f(b)(b-c).$$

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22. Let $f \in \mathcal{R}([a,b])$ and $\epsilon > 0$. Show that there exists a continuous function g on $[a,b]$ such that
$$\int_a^b |f(x) - g(x)|^2 dx < \epsilon.$$

23. Let f be a real and continuously differentiable function on $[a,b]$ such that $f(a) = f(b) = 0$ and
$$\int_a^b |f(x)|^2 dx = 1.$$
Show that
$$\int_a^b x f(x) f'(x) dx = -\frac{1}{2}.$$